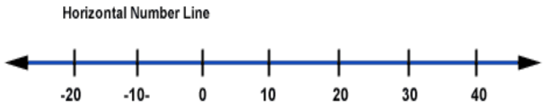
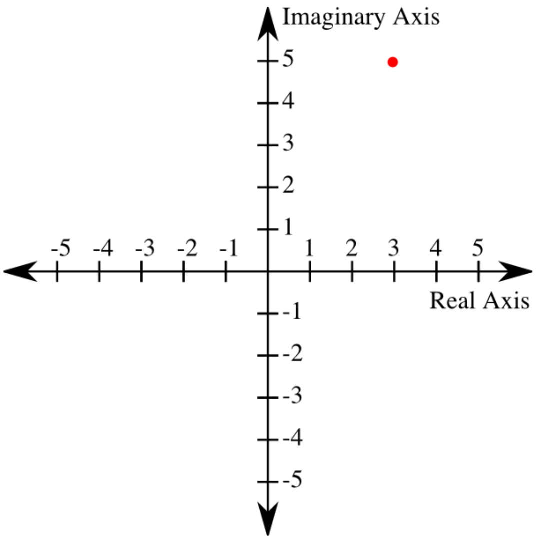
”in English” ☺

Complex Numbers



Numbers we know!!



And the “new” numbers:

Notice: Rectangular and polar !!

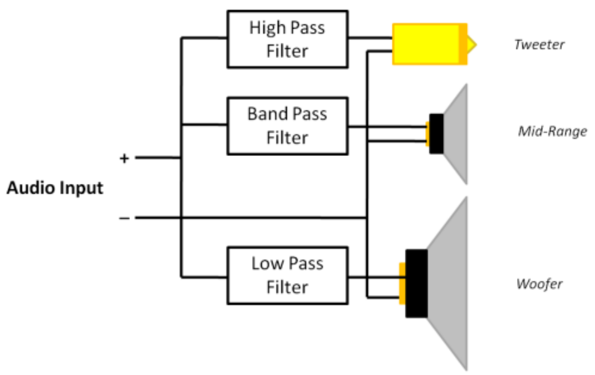
So Why? For what?

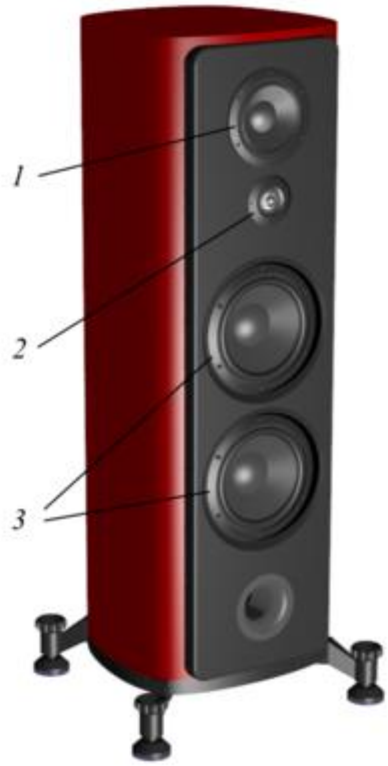
Example. We will examine a circuit

Filter for loudspeaker

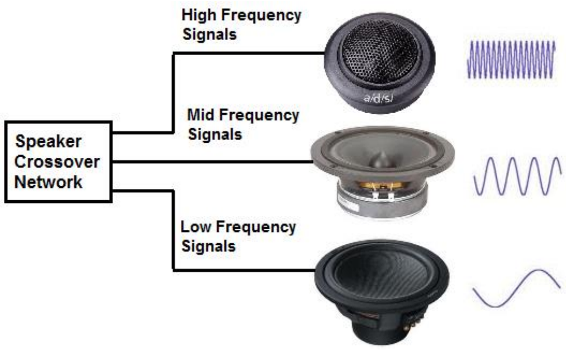
Loudspeaker for home use with three types of drivers

1. Mid-range
2. Tweeter (High frequencies)
3. Woofers (Low frequencies)





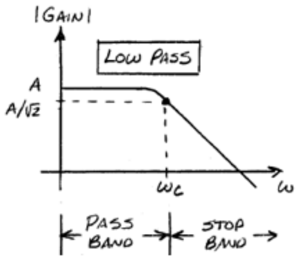
A small cone with a small weight can be accelerated as needed for generating high frequencies !!

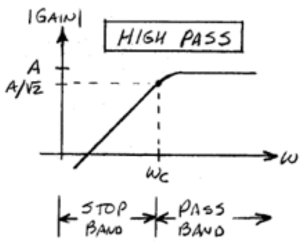


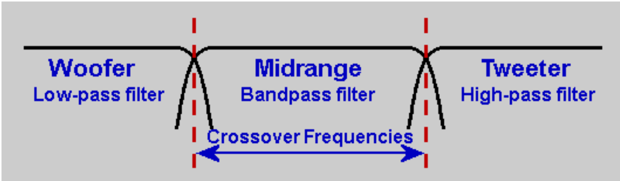
A filter guides low frequencies to the Woofer, high frequencies to the Tweeter, and mid-range frequencies to the Mid-range speaker.

Why? Well the Woofer is so big and heavy, it can´t move so fast. It´s too heavy !!

Electronic filters can be arranged to let low, mid or high frequencies pass from input to output.



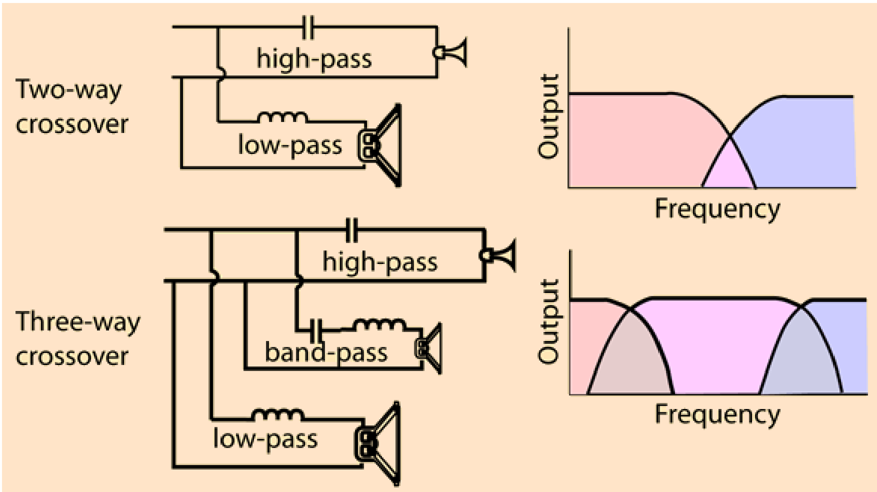


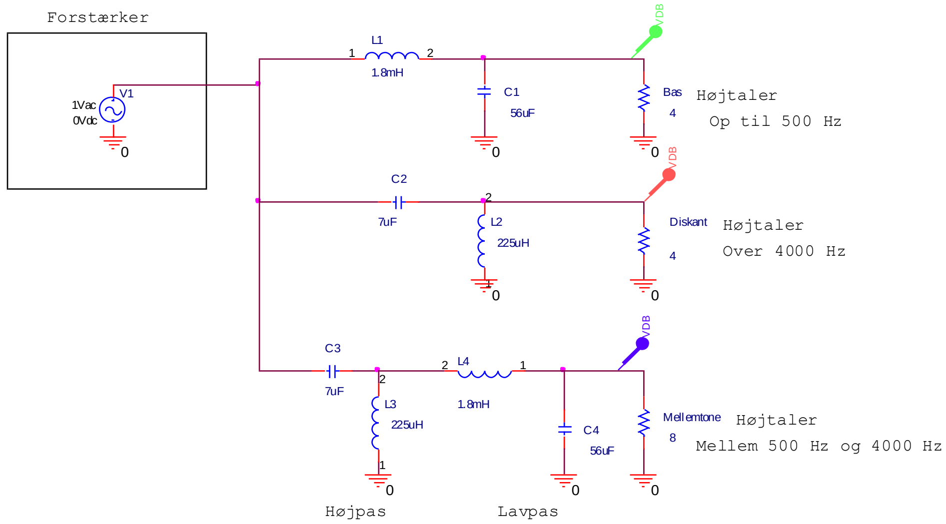


Note that the x-axis shows the frequency

Here´s examples how filters can be built.

There are used capacitors and coils.





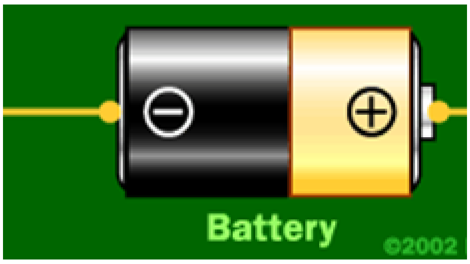
Circuit example:

(Simulation software)

In electronic circuits we find different basic components. **Resistors**, **Capacitors**, **Coils**, Transistors etc.

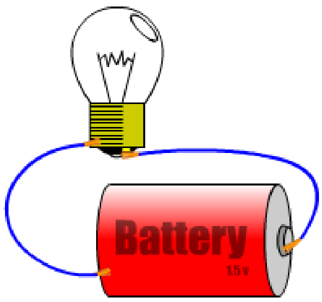
Different components behave different, and even different at different frequencies.

We might first take a short look at a battery:

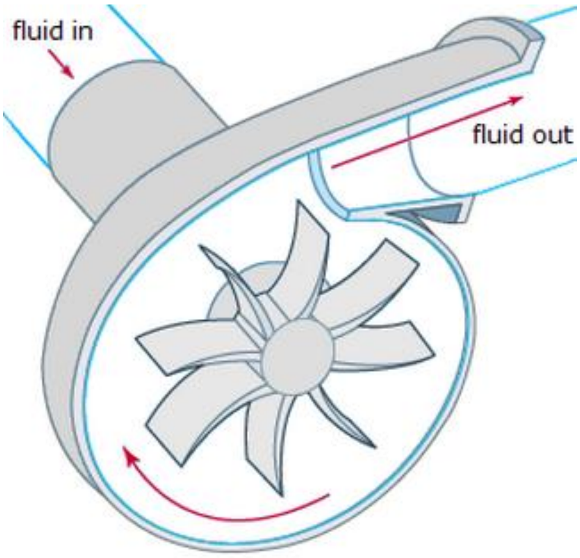


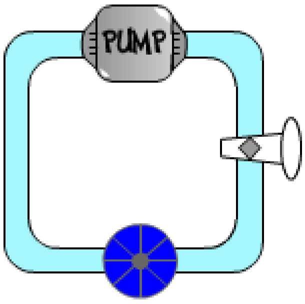
What does a battery really do ??

In a circuit the current is equal all over in the wire.



We can consider the Voltage as an Analogy to water pressure:





The water is pressed up to a higher pressure by the pump. It flows to the water-motor forcing it to turn. Thus transfer energy to the motor.

The water returns to the pump.

http://www.wermac.org/equipment/pumps_centrifugal.html

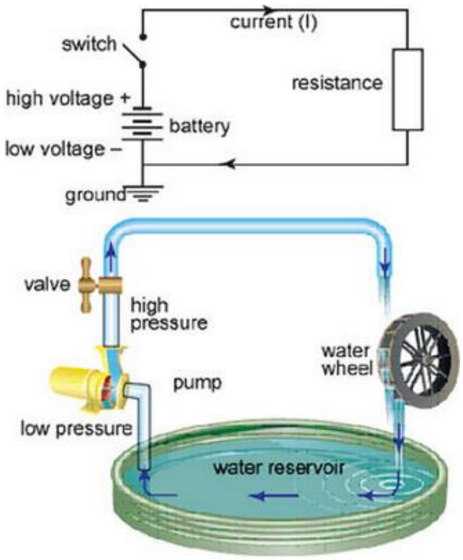
http://www.bgfl.org/bgfl/custom/resources_ftp/client_ftp/ks3/science/elecricity_2/electricity.html

We have an battery giving pressure to electrons.

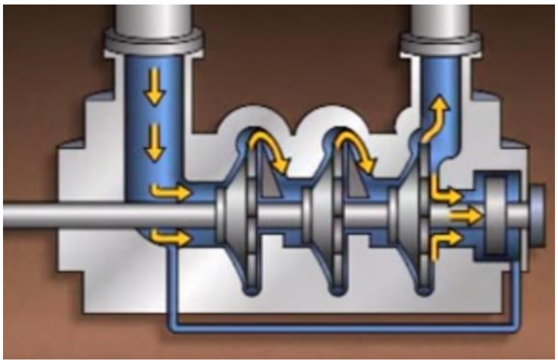
The current flows trough the resistor and heat it up. Delivering energy to it.

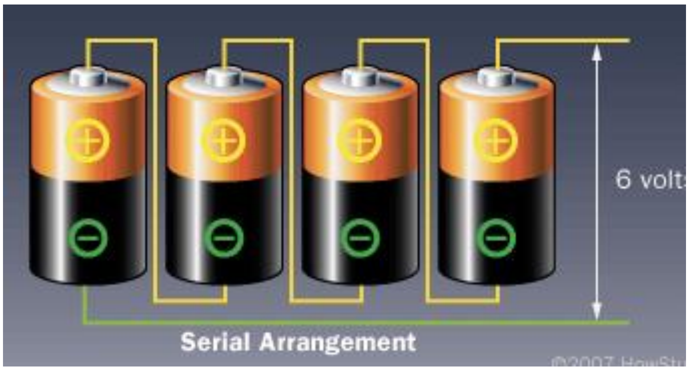
Energy have been tranferred from the battery to the resistor.

Just like the water-motor.



Here´s how we get higher pressure = higher voltage.



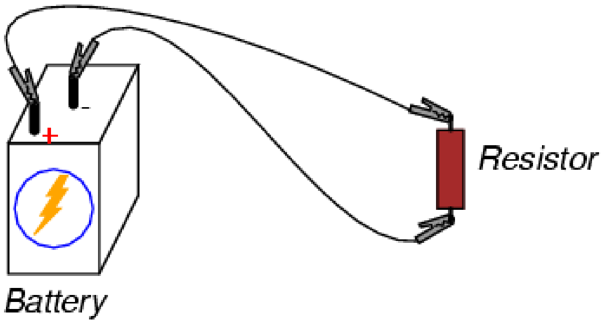


Let’s examine a resistor and a capacitor:

Resistor:

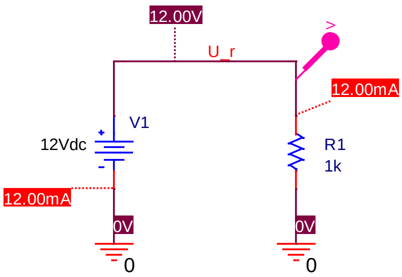
If a voltage generator, (Electron pressure) is applied to a resistor, we can measure a current flowing in the wire.

If the voltage is constant, we can measure the same current over time. The current is constant.

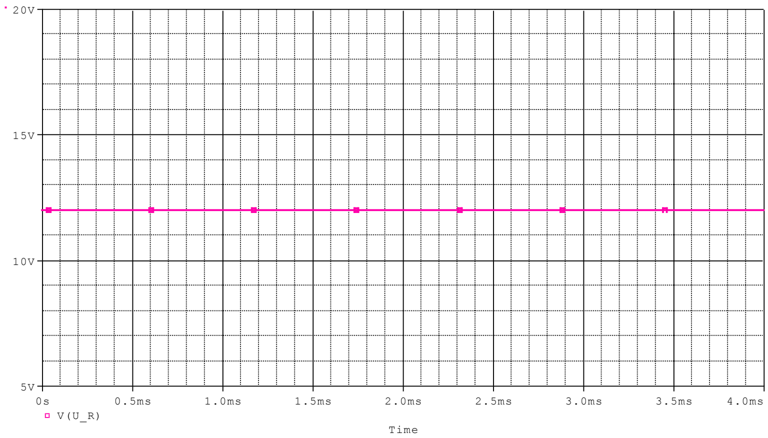


It Follows Ohms Law, $U = I \cdot R$

Here we take a look at some software for simulating circuits. It shows these graphs:



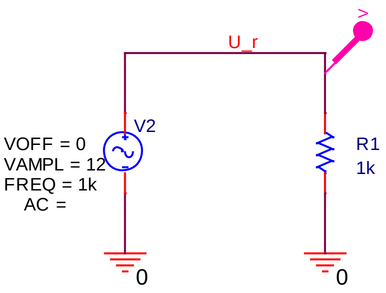
The circuit.

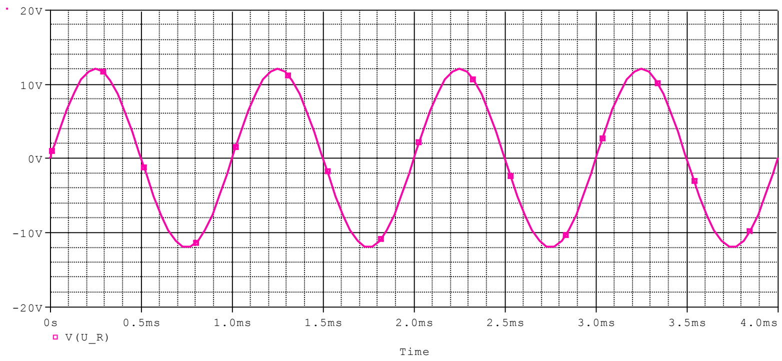


And the graph:

Alternating Voltage

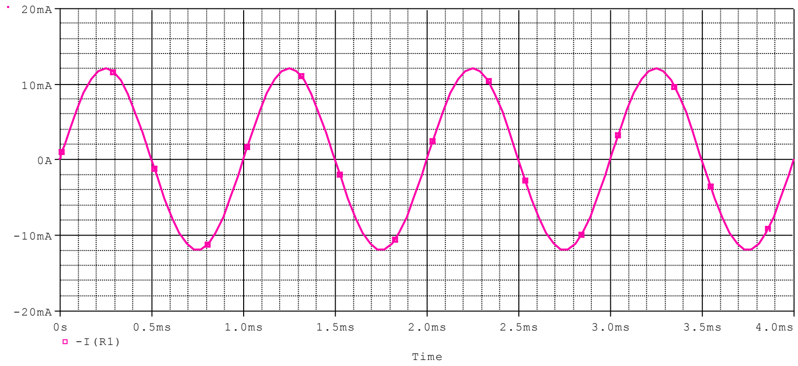
If we now apply an alternating voltage to a resistor, i.e. a sine wave, we get:





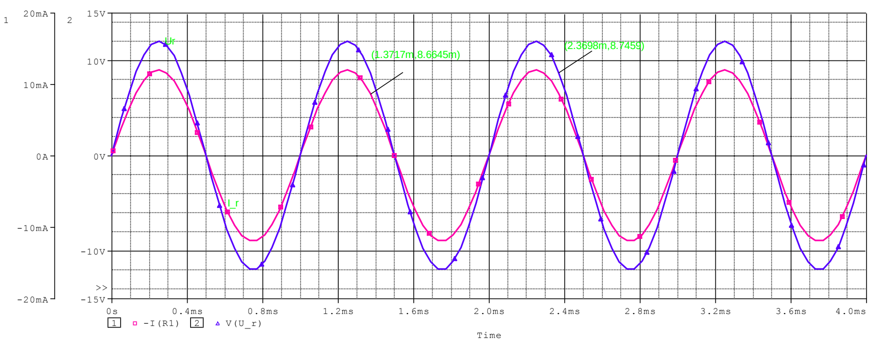
The frequency is here chosen to be 1000 Hz. The peek voltage is set to12 Volt.

A graph for the current looks like this:



And put together:

Obs. ! Different Y-axis!!



We see, that when the voltage is zero, the current also is zero. When the voltage is at its biggest, the current is at its biggest.

Thus:

If a changing voltage is applied directly to a resistor, we find, that the current also changes, - prop to the voltage.

The generator pumps the charges forward and back.

It’s the same electrons that are pushed a bit forward and again back again according to the frequency.

The charges travels only very short distance, less than 1 mm!! But all the electrons pushes to the next, and they push to the next etc.

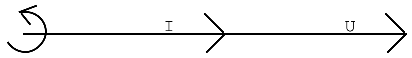
Just like a long train. Each wagon push the next. They all have the same speed. Just moving a bit forward and back !!

Phase angle:

Also we see in the graph above, that the current and voltage are in phase. They are present at the same time!

The phase-angle is = 0.

Drawn with a vector-diagram it´ll look like this:

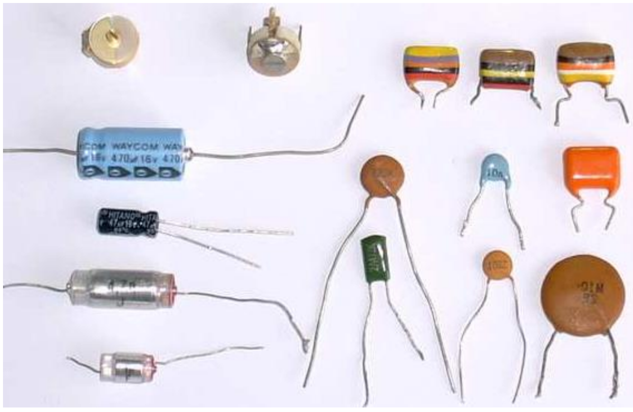


The angle between the voltage and current is called Fi, φ. The angle is 0 degrees.

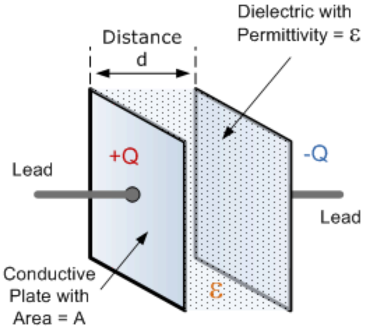
Capacitor

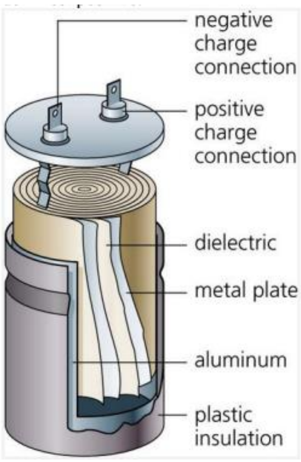
Now let´s examine an capacitor:

Picture of different capacitors:



A capacitor just consists of two conductive plates, insulated from each other.



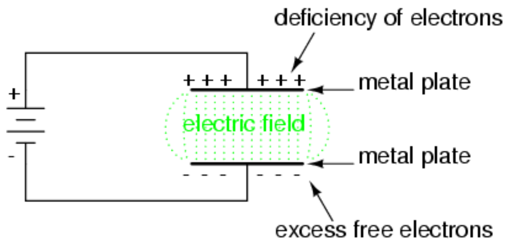


We can perceive a capacitor as a kind of small rechargeable battery.

But with the energy stored in the electrical field between the plates instead of in the chemicals inside the battery.

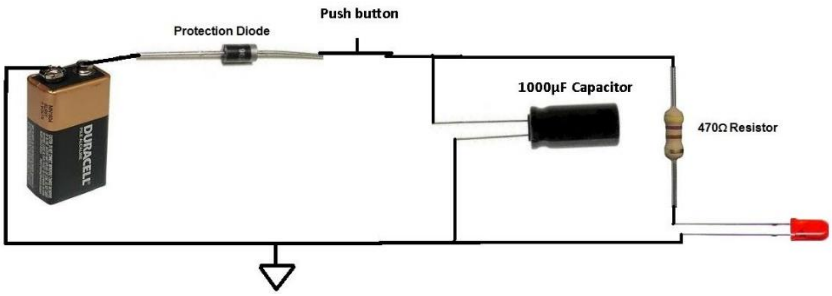
When a capacitor is charged, current flows to the one side, and electrons are stopped there and stored.

But from the other side just as many electrons are leaving the plate. Leaving "holes".



Energy thus is stored in **<u>charge separation</u>**.

If we take the charged battery out of the circuit, the energy remains inside, and can be used later



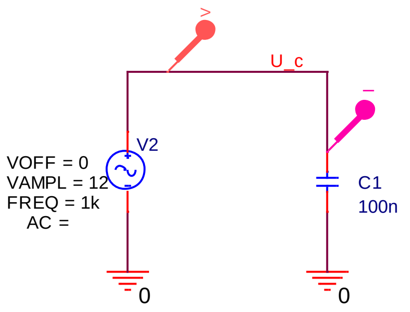
The energy in a capacitor is given by:

$$E = \frac{1}{2} \cdot C \cdot V^2$$

Now a sine voltage generator is applied directly to a capacitor.

The capacitor-voltage will always be the same as the voltage of the voltage-generator.

It means that the charging of the capacitor happens immediately as the applied voltage changes.



The capacitor is charged up and down as the applied voltage changes.

In order to charge a capacitor, electrons are moved, a current is flowing to the capacitor.

If the voltage across the capacitor is changing fast, the current must be bigger.

If the voltage across the capacitor doesn't change up or down, the current must be zero.

This happens when the applied voltage is in its top or in bottom of the sine wave.

Here the slope of the sine voltage is zero. $\frac{dU}{dt} = 0$

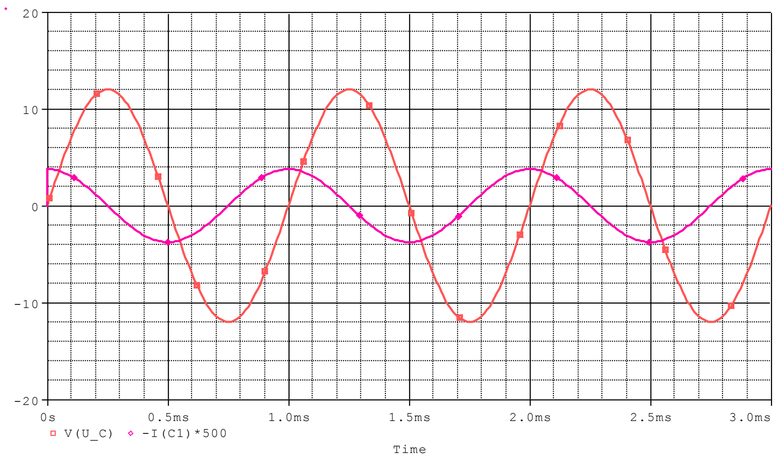
So: At max voltage, $U_{C\text{-max}}$, the current Ic = 0.

And when the voltage crosses zero, its slope is max. $\frac{dU}{dt}\ is\ biggest$

Thus the current will be biggest when the voltage is crossing zero.

On a graph it looks like this:

Just as expected.



Graph for the voltage and the current in a capacitor. The phase angle Φ = 90 degrees

We see, that the current is before the voltage

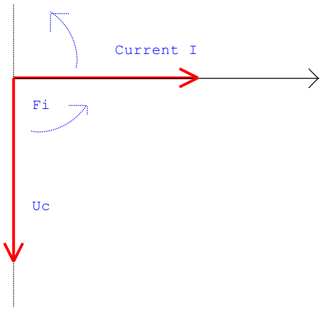
Thus:

In a capacitor, the Current leads the Voltage by 90 degrees.

In a vector diagram it can be drawn like this:

The phase shift is 90 degrees.

Current is 90 degrees in front of the voltage



We might know the phase shift concept from our daily life.

Examples

The highest temperature normally is about a month after midsummer

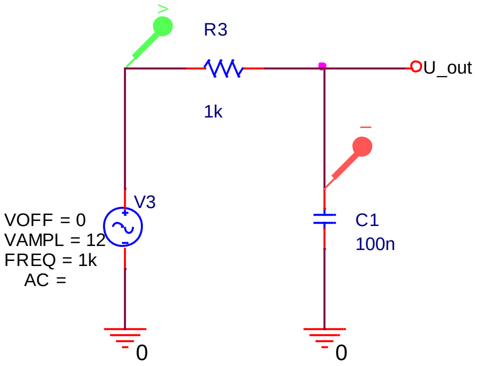
The coldest period is after Dec. 21. It lags the calendar.

Also our day is phase shifted. Our waken period is not centered around noon, when the sun is at its highest.

RC-Circuit

Let’s now apply a sine voltage to a resister in series with a capacitor

Still with 12 Volt and 1 KHz applied.



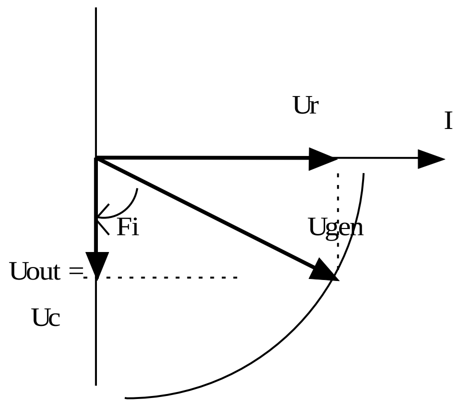
The applied voltage is divided between the resistor and the capacitor.

When the current is flowing, same amount flows in all components. It cannot be stored anywhere.

Like all wagons in a train have same speed!

The current is equal in both components. It´s common to draw it on the X-axis.

From before we know, that the voltage across the resistor is present, when current flows.



Vector diagram for U_R, U_C and U_{GEN} in a RC-circuit.

The current in the capacitor is in front of the voltage, thus the capacitor voltage is drawn downwards.

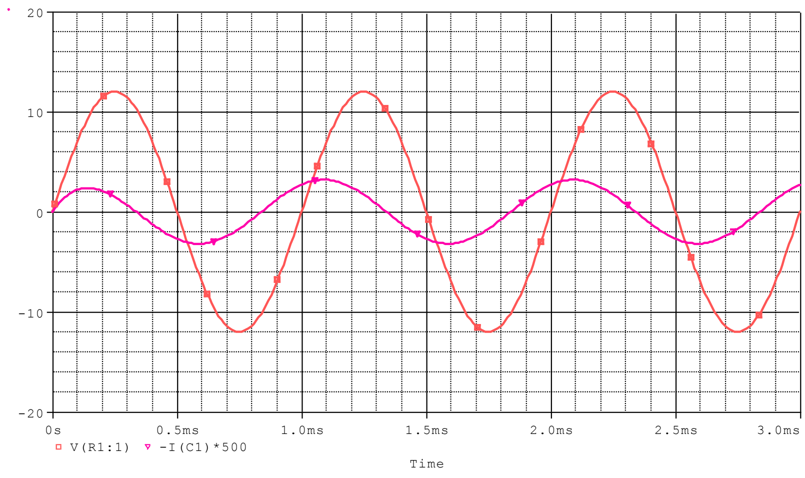
The applied voltage is then found as the geometric sum of U_r and U_c.

They can't be added directly. They have both **value** and **direction**.

In the circuit, U_{OUT} is equal to U_C, and is lagging the generator voltage.

We have a phase angle between $U_{generator}$ and U_{out}. We call it φ, "Fi".

The graph again:

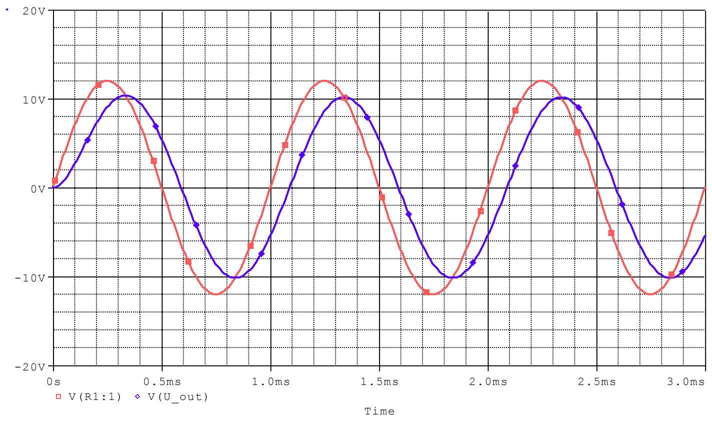


On a graph it looks like this example:

Now the phase between the voltage and current has changed.

And if we examine the output voltage we get these graphs:

The output lags the applied voltage!



The resistance of a capacitor is frequency dependent

In a normal resistor, the resistance always is the same value, no matter the frequency.

But in a capacitor we find, that if we increase the applied frequency, that means charging and discharging, the current has to be bigger in order to charge up and down the capacitor.

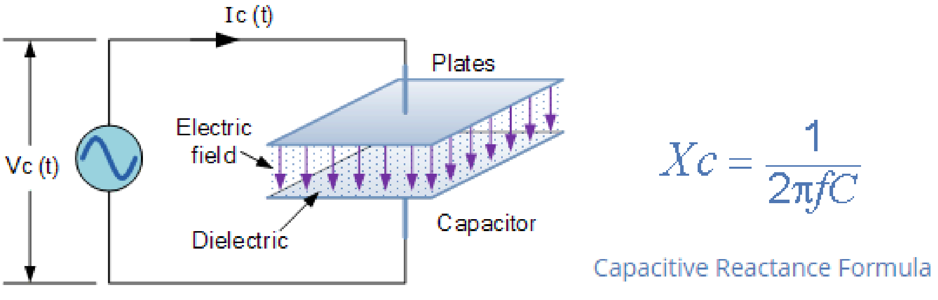
Thus, the more charges can be measured flowing fort and back, the bigger the current must be!

And the bigger current, ***the smaller the resistance in the capacitor seems to be*.**

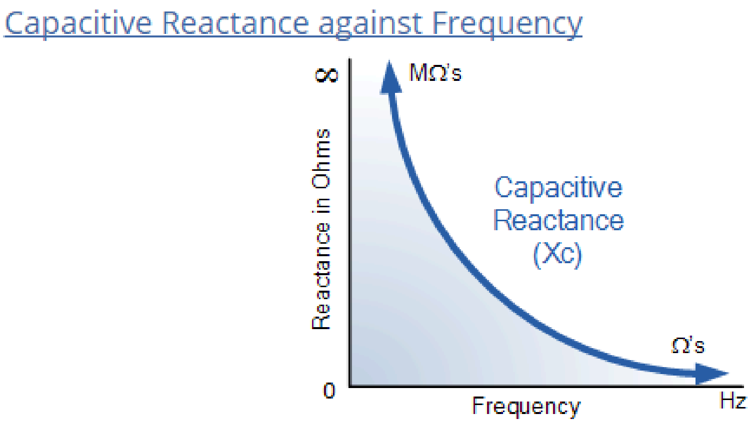
The resistance in a capacitor can be calculated from:

$$X_C = \frac{1}{2 \cdot \pi \cdot f \cdot c}[\Omega]$$

We see the frequency dependency:



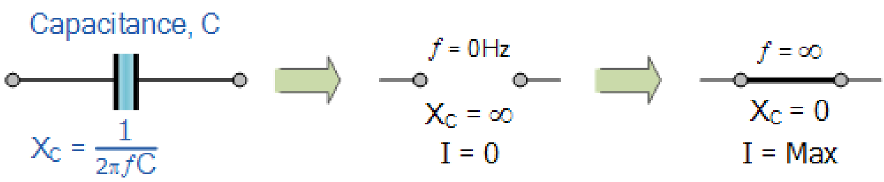
And a graph:



The higher the frequency, the lower the resistance in a capacitor.

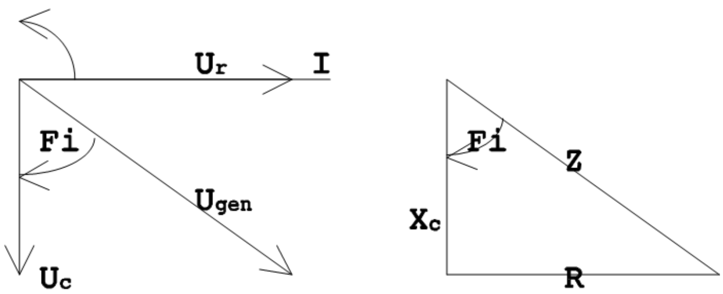
The resistance is called Xc.

Xc at different frequencies



So when the frequency is changed, it has to give an influence on the vector diagram

We'll see more later



And the phase angle:

U_C is behind $U_{Generator}$. The angle "fi" gives the phase angle. The angle between U_{gen} and U_{Out} can be calculated by:

$$\text{Tangent "fi"} = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{U_R}{U_C} = \frac{R}{X_C}$$

$$\text{"fi" then is} \quad tg^{-1}\left(\frac{R}{X_C}\right)$$

Obs. Frequency dependent because Xc is frequency dependent!

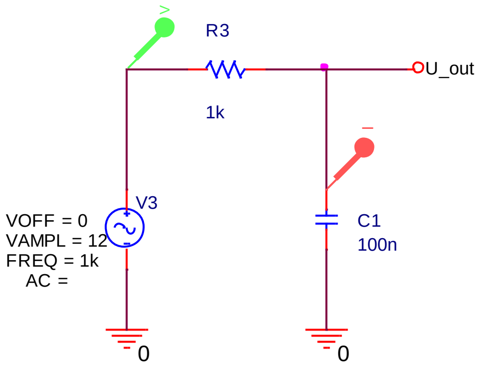
The RC-circuit seen as a voltage divider

Again have a look at the RC circuit.

It is a voltage divider

It’s also called a “Low Pass” circuit.

Why?

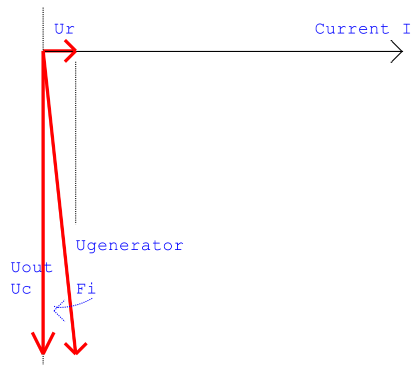


First we look **at low frequencies**

The capacitor don´t steel much of the signal

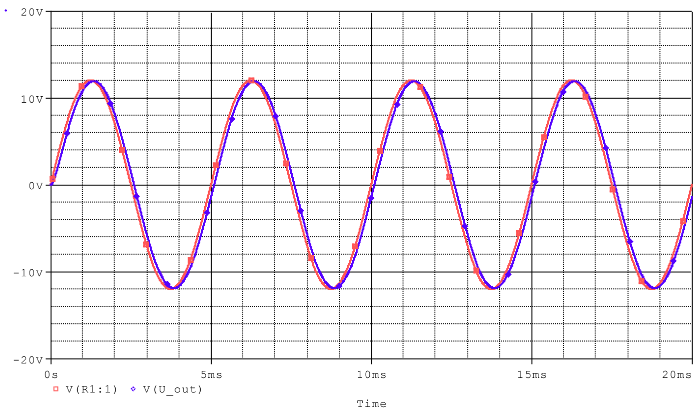
There´s time enough to charge and discharge the capacitor.

Thus: U_{Out} is nearly equally to U_{In} at very low frequencies.



Here a frequency of 200 Hz is applied.

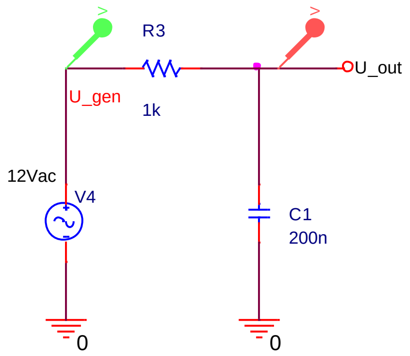
We see the output voltage is close to the input voltage



At high frequencies

Xc will be low, nearly a short circuit. The capacitor attenuates or steels the signal, U_{out} is very low.

There isn’t time enough to charge totally up and down for each cycle

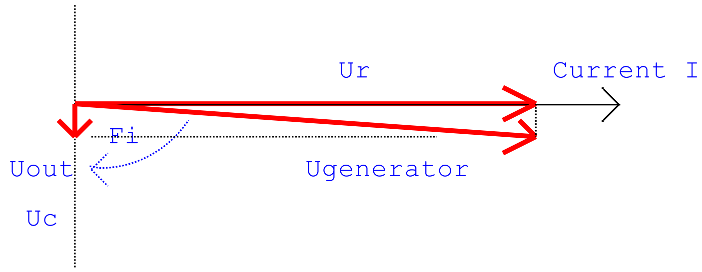


Thus the voltage across the capacitor will not change much.

I.e. At high frequencies the capacitor will be nearly a short circuit, because the Xc is very small

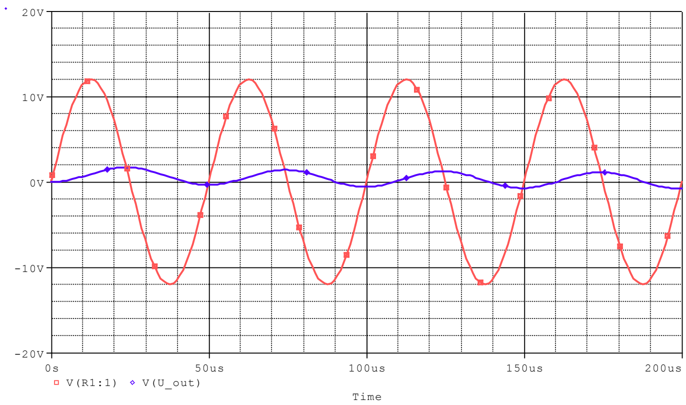
At high frequencies the U_{Out} – signal is attenuated!

In a vector diagram it will look like this:



Here 20 KHz is applied.

The output voltage is very low!

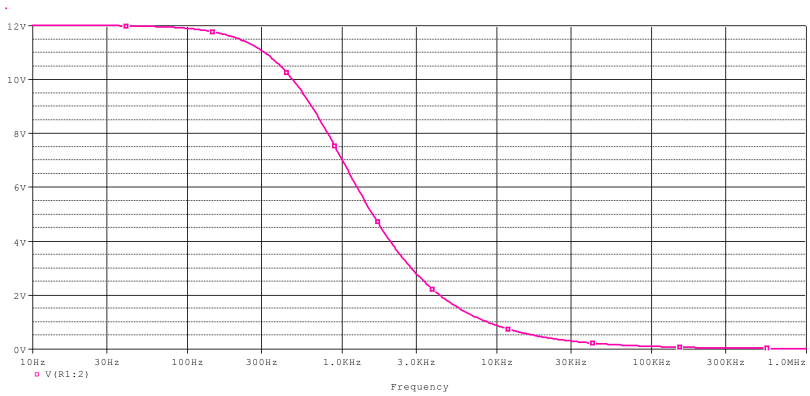


A graph for a frequency sweep can show the output voltage like this:

Graph for " all " frequencies:
From 10 Hz to 1 MHz.

At low frequencies the output voltage is equally to the input voltage.

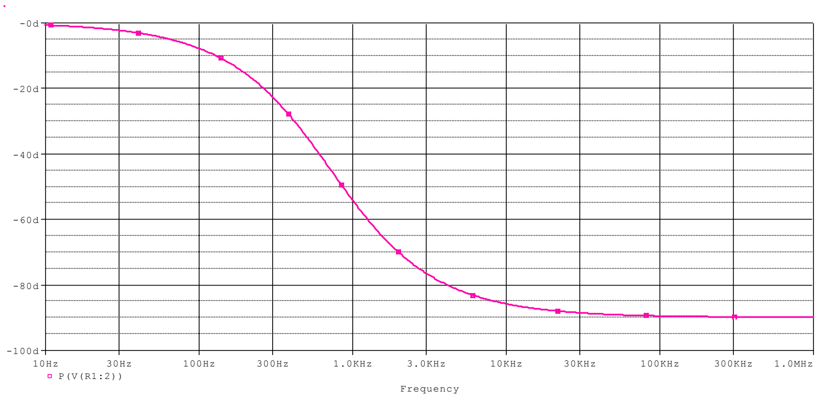
They can " pass " the circuit.



And we see, for high frequencies, the output voltage is very low.

This gives the name to the circuit: Low Pass Filter. Low frequencies passes, the high frequencies is attenuated by the filter.

And the phase angle:



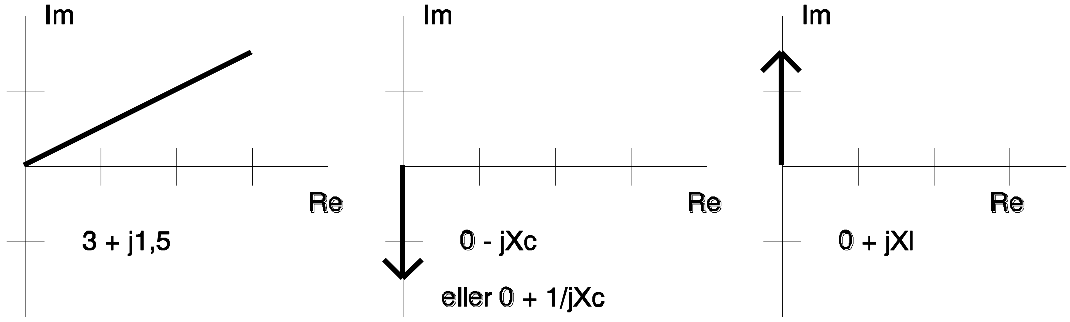
Use of complex numbers.

If now we want to use math to describe the graphs for output Voltage and Phase-angle as function of the frequencies, and we know, the vectors has different directions, we might get a bit stuck.

But here we can use complex numbers to describe the vectors, or the phase angle between voltage and current.

We find:

A random vector, Capacitor Coil



In electronic ‘j’ is used instead of ‘i’ – for describing the imaginary axis!!

The j tells us that the vector is 90 degrees out of phase.

+j is upwards, -j is downwards

Resistor:

A complex notation for a resistance in a resistor will be:

$$Z_R = R + j0$$

"j0" tels, that the resistor has no imaginary part, thus pure reel.

Capacitor:

For the capacitor, in Complex notation we find, that

$$Zc = 0 - jXc$$

The vector starts in Origo, and the minus tells, it´s pointing downwards

Previously we found the resistance in a capacitor to be calculated from:

$$X_C = \frac{1}{2 \cdot \pi \cdot f \cdot c}[\Omega]$$

So we get:

$$Zc = 0 - j\frac{1}{2\pi fC}$$

And, because $2\pi f C$ also can be written as ωC, (omega * C):

$$Zc = 0 - j\frac{1}{\omega C}$$

Or, because

$$-j\ \frac{1}{\omega C} = -\frac{j}{\omega C} = -\frac{j \cdot j}{j \cdot \omega C} = \frac{1}{j\omega C}$$

$$Zc = 0 + \frac{1}{j\omega C}$$

Notice that j*j = -1 !

But why is $j^2 = -1$??

The complex vector j can be described as 0 + j1. Meaning, 0 along the x-axis, and 1 upwards

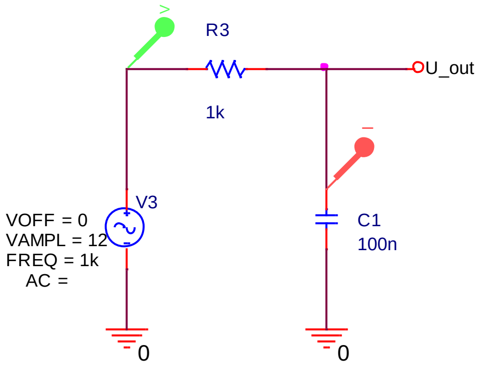
In polar form it equals 0 + j1 = 1 ∠90

So j * j equals (1∠90) * (1∠90). This equals 1*1∠(90 + 90) = 1∠180.

1∠180 is the same as –1.

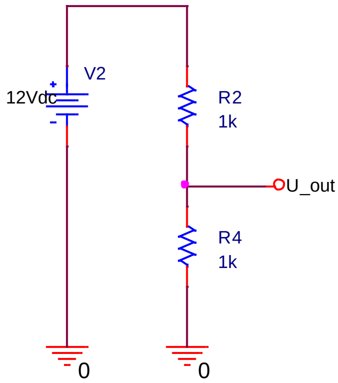
Again we look at the RC circuit.

Also called a low pass filter



R and C makes a voltage divider.

Like this with 2 resistors:



Uout equals the applied voltage, times R4 , divided with the sum of resistors

$$Uout = Uin \cdot \frac{R4}{R2 + R4}$$

The fraction $\frac{R_4}{R_2+R_4}$ can be considered as a “gain”. Less than one, but anyway!

Thus for the RC-circuit we have:

$$Gain = A` = \frac{\overline{X_C}}{\overline{R} + \overline{X_C}}$$

We find:

$$A` = \frac{\overline{X_C}}{\overline{R} + \overline{X_C}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega CR}$$

We need to get rid of the "j" in the denominator. So we multiply with the complex conjugated.

$$A` = \frac{1}{1 + j\omega CR} * \frac{1 - j\omega CR}{1 - j\omega CR} = \frac{1 - j\omega CR}{1^2 - j\omega CR + j\omega CR - j^2(\omega CR)^2}$$

The two middle terms in the D disappear. And j*j is equal -1. Thus we get:

$$A` = \frac{1 - j\omega CR}{1^2 + (\omega CR)^2}$$

This equation describes the gain at all frequencies!

It consists of a **reel part** and an **imaginary part**.

The terms with a "j" describes vertically to the x-axis.

The equation can be divided up into a reel part, without "j", and an imaginary part, with "j" in front.

The denominator is common.

$$A' = \frac{1}{1^2 + (\omega CR)^2} \cdot (1 - j\omega CR)$$

Or

$$A` = \frac{1}{1^2 + (\omega CR)^2} - j\frac{\omega CR}{1^2 + (\omega CR)^2}$$

The length of the sum of the real and the imaginary vectors is calculated as:

$$A` = \sqrt{\left(\frac{1}{1 + (\omega CR)^2}\right)^2 + \left(\frac{\omega CR}{1 + (\omega CR)^2}\right)^2}$$

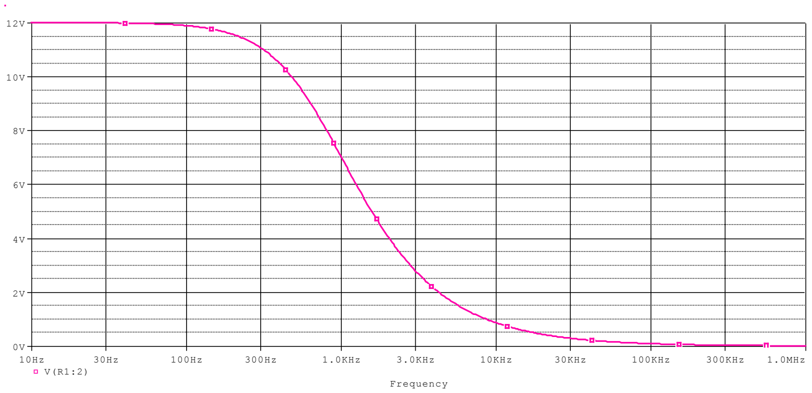
And the phase angle $$\varphi = tg^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) \quad or \quad \varphi = tg^{-1}\left(\frac{-\omega CR}{1}\right)$$

The Simulating software – ORCAD - uses complex numbers to calculate graphs for circuits. Here a frequency sweep is applied.

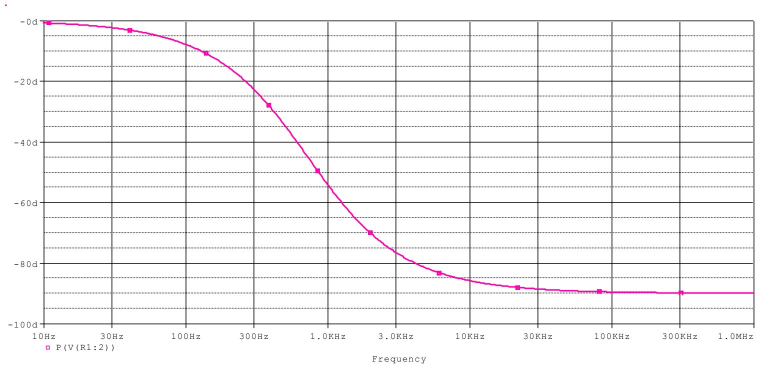
Using the simulating software we get:

This upper graph shows the output voltage

And below we see the phase



The Phase.



Test for very low frequencies frequency → 0, and thus also ω → 0

We had the equation:

$$A` = \sqrt{\left(\frac{1}{1+(\omega CR)^2}\right)^2 + \left(\frac{\omega CR}{1+(\omega CR)^2}\right)^2}$$

And the phase angle $$\varphi = tg^{-1}\left(\frac{-\omega CR}{1}\right)$$

We get for ω → 0

$$A` \rightarrow \sqrt{\left(\frac{1}{1+0^2}\right)^2+\left(\frac{0}{1+0}\right)^2} \angle tg^{-1}\left(-\frac{0}{1}\right)$$

$$A` \rightarrow \sqrt{1^2} \angle tg^{-1}(-0) \rightarrow 1\angle 0$$

Telling us, that at very low frequencies $U_{out} = U_{in}$ multiplied with $1\angle 0$

That means, that $U_{out} \rightarrow U_{in}$ and with "0" degrees phase angle.

It also is the result of a logically view of the circuit. The capacitor does not cause any load for very low frequencies.

Test for very high frequencies

Now an examination for f → infinite. Meaning also ω → infinite:

Again we had:

$$A` = \sqrt{\left(\frac{1}{1+(\omega CR)^2}\right)^2+\left(\frac{\omega CR}{1+(\omega CR)^2}\right)^2}$$

And the phase angle $\varphi = tg^{-1}\left(\frac{-\omega CR}{1}\right)$

Giving us for ω → infinite:

$$A` \rightarrow \sqrt{\left(\frac{1}{1+\infty^2}\right)^2+\left(\frac{\infty}{1+\infty^2}\right)^2} \angle tg^{-1}\left(\frac{-\infty}{1}\right)$$

$$A' \rightarrow \sqrt{\frac{1}{\infty^4}+\frac{1}{\infty^2}} \angle -90$$

$$A' \rightarrow 0\angle -90$$

So, at very high frequencies $U_{out} = U_{in}$ times $0\angle -90$. I.e. $U_{out} = U_{in}$ times 0 and "90" degrees behind.

The output amplitude → "0", act as a short circuit, and the phase angle is -90 degrees.

Finally Test for the frequency → f_0

If the frequency → f_0, i.e. the corner frequency in the Bode Plot, - or also the frequency, where **the size of** R = Xc we find:

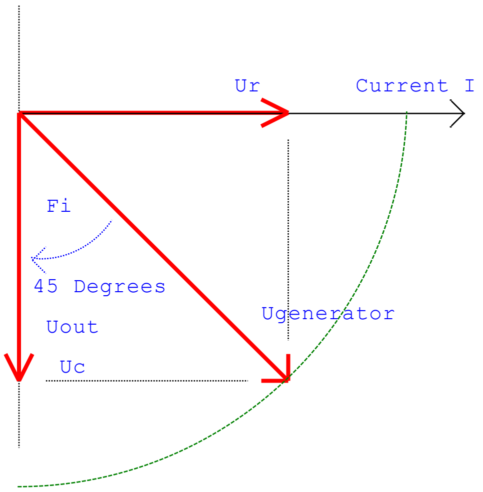
$$R = Xc \Leftrightarrow R = \frac{1}{\omega C} \Leftrightarrow \omega CR = 1$$

$$A` \to \sqrt{\left(\frac{1}{1+1^2}\right)^2 + \left(\frac{1}{1+1^2}\right)^2} \; tg^{-1} - \left(\frac{1}{1}\right)$$

$$A` \to \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \angle -45$$

$$A` \to \sqrt{\frac{1}{4}+\frac{1}{4}} \angle -45 \to \sqrt{\frac{2}{4}} \angle -45 \to \frac{\sqrt{2}}{2} \angle -45$$

$$A` \to 0{,}707 \angle -45$$



$$U_{Out} = \frac{U_{In}}{Cos(45)} = \frac{U_{In}}{\sqrt{2}} = Uin \cdot 0{,}707$$

A` = 0,707

Phase angle is - 45 degrees

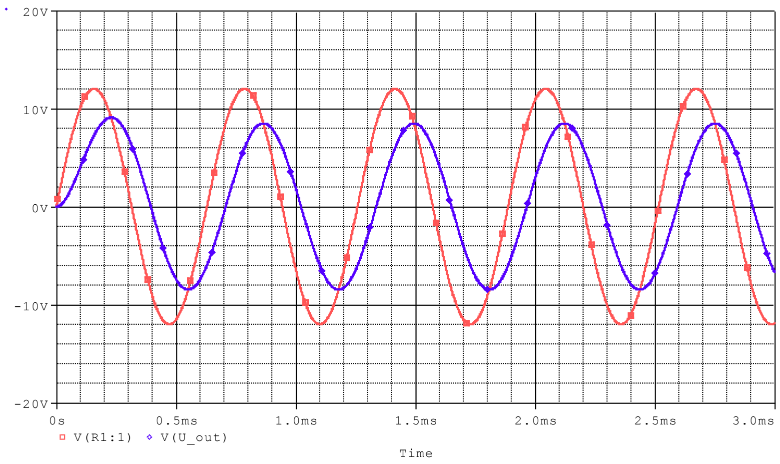
(at the corner frequency f0)

Here the output is equal to the applied voltage times 0,707.

The frequency applied is 1590 Hz.

It´s where the value of the resistor equals Xc.

Thus: $1K = \frac{1}{2 \cdot \pi \cdot f \cdot c}$



The following shows a sketch of the Bode Plot and the phase angle: